SECULAR LONG PERIODIC AND HIGH FREQUENCY SEMI MAJOR AXIS TIME RATES OF CHANGE DUE TO EMISSION OF GRAVITATIONAL RADIATION

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Abstract - For a two-body system, we derive expressions for the secular, long periodic and high frequency effects caused on to the semi-major axis of the secondary body due to the emission of gravitational waves. This is possible by considering the time rate of change of the total energy of the system, and substituting for the energy loss due the emission of gravitational waves. Next, we convert the time rate of change of the semimajor axis as a function of the orbital elements using standard a celestial mechanics approach, where the inverse powers or the orbital radial vector r is written as a function of the orbital elements with the help of the eccentricity functions G(e). We apply our result to Mercury/Sun system, and the secular time rate of change of the semi-major axis has been found to be the order -1.660×10^{-12} m/y. In the case where the primary is a super massive black hole (SBH) the secular semimajor axis time rate of change of the secondary of eccentricity 0.9 is equal to -23 m/y. In the case of Mercury/Sun, this secular change is an extremely small and it cannot be easily detected by today's Earth based technology. The secular rate change related to the SBH, can be probably detected by a future space technology. Therefore, such two-body system where the primary has mass equal to that of the Sun does not really constitute a good candidate for observing the quadrupole radiation effects on the semimajor of the orbital body. On the other hand, SBH will constitute better candidates if new space-based technologies are to be in effect in the nearest future.

I. INTRODUCTION

One of the most interesting predictions of Einstein's general theory of relativity and its field equations is that associated with the existence of gravitational waves. The experimental detection of the gravitational radiation is extremely difficult due to the weak coupling between matter and gravitation. A simple periodic source of gravitational waves is the rotating quadrupole. gives the total energy radiated by such a system to be (Ohanian 1994):

$$\frac{dE}{dt} = -\frac{G}{45c^5}\dddot{Q}^{\mu\nu}\dddot{Q}^{\mu\nu}. \quad (1)$$

where $\ddot{Q}^{\mu\nu}$ is the contravariant quadrupole moment tensor components defined as follows (ibid, 1994):

$$Q^{\mu\nu} = \int \left(3x^{\mu}x^{\nu} - r^2\delta^{\mu}_{\nu}\right)\rho(\vec{x})dx^3. \quad (2)$$

where x^{μ}, x^{ν} are the contravariant vector components, $\rho(\vec{x})$ material density function.

Assuming a planet of mass m orbiting in a circular orbit of radius r around a star of mass where M >> m, we can write the time rate of change of the energy radiated by this system is given (Peters and Mathews, 1963) by:

$$\frac{dE}{dt} = -\frac{32G^4M^2m^2(m+M)}{5(1-e^2)^{7/2}c^5a^5}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \quad (3)$$

The goal of this contribution is to derive the secular, long periodic, and short periodic effects caused on to semimajor axis by emission of gravitational waves of a two-body system, and examine if such effects are detectable by today's technology. For that, we start with the expression of the total energy of the system. Next, we find an expression of the time rate of the total energy of the system we substitute for the known expression from Eq. (3). Finally, we obtain and expression for the rate of the semimajor axis, in which the inverse radial powers of the radial distance are, expressed as function of the orbital elements with the help of the eccentricity functions, and trigonometric arguments of different indices that result from its spherical harmonic expansion.

II. THE SEMIMAJOR AXIS EFFECT

Starting with the total energy per unit mass of such a system we write:

$$E = -\frac{GMm}{2a}, \quad (4)$$

taking the time derivative of Eq. (4) we obtain:

$$\frac{dE}{dt} = \frac{GMm}{2a^2}\frac{da}{dt}, \quad (5)$$

assuming M >> m using Eq. (3) in Eq. (5) and solving for $\dot{a}$ we have:

$$\frac{da}{dt} = -\frac{64G^3M^2m}{5c^5a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \quad (6)$$

Next the term $1/r^{\ell+1}$, where ℓ is integer (see below), can be written as a function of the eccentricity functions $G_{\ell pq}(e)$ and the orbital elements of secondary body in the apparent right ascension system as follows [Kaula, 2000; p.35]:

$$\frac{1}{r^{\ell+1}}\begin{pmatrix}\cos\\ \sin\end{pmatrix}[(\ell-2p)(\omega+f)+m(\Omega-\Theta)]=\frac{1}{a^{\ell+1}}\sum_{q=-\infty}^{+\infty}G_{\ell pq}(e)\begin{pmatrix}\cos\\ \sin\end{pmatrix}[(\ell-2p)\omega+(\ell-2p+q)M+m(\Omega-\Theta)]$$

, (7)

where f is the true anomaly, Θ is the Greenwich sidereal time, ℓ is the degree and m is the order of the spherical harmonic expansion of the potential, (p, q)∈ Z and $0 \le p \le \ell$. The indices ℓ, p, q, m identify the eccentricity function and the trigonometric argument associated with a particular spherical harmonic term of degree ℓ and order m. These terms arise from the potential of the primary when it is expressed in terms of spherical harmonics as given in Kaula (cf. Eq. (1.31); Kaula, 2000). Therefore, for a non-circular orbit $\ell = 2$, for 1/a^3 which can be written as follows:

$$\frac{1}{r^3}=\frac{1}{a^3}\sum_{q=-\infty}^{+\infty}G_{2pq}(e)\cos[(\ell-2p)\omega+(\ell-2p+q)M+m(\Omega-\Theta)] \quad (8)$$

and therefore Eq. (8) becomes

$$\frac{1}{a^3}=-\frac{64G^3M^2m}{5c^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\frac{1}{a^3}\sum_{q=-\infty}^{+\infty}G_{2pq}(e)\cos\begin{bmatrix}(\ell-2p)\omega+(\ell-2p+q)M\\ +m(\Omega-\Theta)\end{bmatrix}$$

. (9)

III. THE SECULAR SEMIMAJOR AXIS EFFECT

Next, we examine only the secular terms resulting from the RHS of Eq (6). We can do this by eliminating the low frequency term from Eqs. (7) by setting $\ell-2p=0$. Similarly, from Eq. (9) and, we eliminate the terms that are varying with high frequency, i.e., the terms that are functions of the mean anomaly M, and $(\Omega-\Theta)$. This can be achieved by setting their respective coefficients to zero, which results in $(\ell-2p+q)=0$, and m = 0, which imply q = 0 since $\ell=2p$. To proceed with the calculation of the secular time rates of change of the orbital elements due to Eq. (6), we need to calculate of eccentricity function $G_{\ell pq}(e)$ is not a trivial process because it requires the use of the so called Hansen coefficients $X_k^{n,m}$. Following Giacaglia, (1976) we have that:

$$G_{\ell pq}(e)=X_{\ell-2p+q}^{-(\ell+1),(\ell-2p)}, \quad (10)$$

and therefore

$$G_{210}(e)=X_0^{-3,0}=(1-e^2)^{-3/2}. \quad (11)$$

These conditions must hold simultaneously and finally, Eqs. (6) takes the form:

$$\frac{da}{dt}=-\frac{64G^3M^2m}{5c^5(1-e^2)^{7/2}a^3}G_{210}(e)\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right),$$

(12)

which finally becomes:

$$\frac{da}{dt}=-\frac{64G^3M^2m}{5c^5(1-e^2)^5a^3}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right), \quad (13)$$

which in the case of a circular orbit e = 0 simplifies to:

$$\frac{da}{dt}=-\frac{64G^3M^2m}{5c^5a^3}, \quad (14)$$

which also results from Eq. (6) when r = a.

IV. THE LOW FREQUENCY SEMIMAJOR AXIS EFFECT

Focusing on the low frequency terms of Eq. (6), we eliminate the terms from Eq. (9) that vary with high frequency. This can be achieved by setting their respective coefficients to zero resulting to $\ell-2p+q=0$ and m = 0. For the $1/a^3$ term in (8) we have that $\ell=2$ and $0\le p\le\ell$ and $q=2p-\ell$, which implies that $q\in[-2,\ 2]$ therefore, using Eq. (9) Eq. (6) and summing over p becomes:

$$\frac{da}{dt}=-\frac{64G^3M^2m}{5c^5a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\sum_{p=0}^{2}G_{2p2p-2}\cos[(2-2p)\omega]$$

. (15)

Carrying out the summation in the above equations, we obtain:

$$\frac{da}{dt}=-\frac{64G^3M^2m}{5c^5a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)[(G_{20-2}(e)+G_{222}(e))\cos 2\omega+G_{210}(e)]$$

, (16)where the eccentricity functions in Eq. (16) have the following values

$$G_{222}(e) = G_{20-2}(e) = 0 \quad , \qquad (17)$$

$$G_{210}(e) = \left(1-e^2\right)^{-3/2} \quad , \qquad (18)$$

and therefore the final equation for the low frequency effects becomes:

$$\frac{da}{dt} = -\frac{64G^3M^2m}{5c^5a^3\left(1-e^2\right)^5}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \qquad (19)$$

where for circular orbit e = 0 Eq. (19) becomes

$$\frac{da}{dt} = -\frac{64G^3M^2m}{5c^5a^3}. \qquad (20)$$

V. HIGH-FREQUENCY SEMIMAJOR AXIS EFFECT

In order to obtain the high frequency components of Eq. (6), we simply eliminate the secular and low-frequency terms in (9) and we obtain:

$$\frac{da}{dt} = -\frac{64G^3M^2m}{5c^5a^3\left(1-e^2\right)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\sum_{m=0}^{2}\sum_{q=1}^{4}G_{21q}(e)\cos[qM+m(\Omega-\Theta)] \quad . \qquad (21)$$

We proceed with the derivation of the high frequency effects by summing over index q ≤ 4. When q > 4 there is no contribution because the resulting terms are equal to zero since they are multiplied with eccentricity functions that they are equal to zero. Therefore the semimajor axis time rate of change due to high frequency terms, as resulting from the emission of gravitational radiation becomes, written in orders of eccentricity up to O(e^4) included:

$$\frac{da}{dt} = -\frac{64G^3M^2m}{5c^5a^3\left(1-e^2\right)^{7/2}}f(e)\left[\begin{array}{l} G_{211}(e)\left(\sin M+\sin(\Theta-\Omega-M)\sin\left(2(\Omega-\Theta)\right)+M\right)+\sin\left(3(\Omega-\Theta)+M\right) \\ +\sin\left(4\left((\Omega-\Theta)+M\right)\right)-G_{212}(e)\left(\begin{array}{l}\sin\left((\Omega-\Theta)+2M\right)+\sin 2M \\ +\sin\left(2M+2(\Omega-\Theta)\right)+\sin\left(2M+3(\Omega-\Theta)\right) \\ +\sin\left(2M+4(\Omega-\Theta)\right)\end{array}\right)\end{array}\right]$$

, (22)

Substituting for the eccentricity functions and simplifying we obtain the following equation:

$$\frac{da}{dt} = -\frac{64G^3M^2m}{5c^5a^3\left(1-e^2\right)^{7/2}}f(e)\left[\frac{e}{16}\begin{pmatrix}1+2\cos(2\Theta-2\Omega) \\ +2\cos(\Theta-\Omega)\end{pmatrix}\left[\left(-\left(24+36e+27e^2+28e^3\right)\sin\left(2\Theta-2\Omega-2M\right)\right)\right]\right] \qquad (23)$$

where again

$$f(e) = \left(1+\frac{73e^2}{24}+\frac{37e^4}{96}\right). \qquad (24)$$

is the original enhancement eccentricity factor that appears in Eq. (3)

VI. ORBITAL DECAY TIMES

We can now calculate the orbital decay time due to secular/maximum low frequency effects using Eq. (19). Integrating this equation and between a_i and a_f we obtain, and imposing that $a_f = 0$ we obtain that

$$t_{D_{\sec}} = \frac{5c^5a_0^4}{256G^3mM^2}\frac{\left(1-e^2\right)^5}{\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}. \qquad (25)$$

The same equation also holds for the in phase long frequency effects. Similarly, using Eq. (23) for the high frequency effects we obtain the maximum decaying time corresponding to in phase effects to be:

$$t_{D_{HF}} = \frac{5c^5a_0^4}{256G^3mM^2}\frac{\left(1-e^2\right)^{7/2}}{\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\left(\frac{9}{2}e+\frac{27}{4}e^2+\frac{81}{16}e^3+\frac{21}{4}e^4\right)}.$$

(26)

VII. NUMERICAL RESULTS

For our numerical calculations, we choose Mercury orbiting around the Sun. Mercury has the following orbital parameters e = 0.205631752, a = 57909083 km. $m = 3.3022\times10^{23}$ kg, and the mass of the sun is $M = 1.99\times10^{30}$ kg (Vallado, 2007). Also the argument of the perihelion $\omega = 77.45645^{\circ}$ (Murray and Dermott, 1999).

First, we proceed with the calculation of the secular effect on the semimajor axis in meters per Earth years. Substituting in Eq. (13) we obtain that:

$$\frac{da}{dt} = -4.652\times10^{-13}\ \text{m/y}, \tag{25}$$

where in the case of a circular orbit having identical orbital parameters to those of Mercury we obtain -3.320×10^{-14} m/a. For the shake of an "extreme" secular semimajor axis change calculation due to gravitational radiation assume a secondary body that orbits a massive black hole (SBH) of the following parameters: mass $M = (3.95\pm0.06)\ M_{solar}$, semimajor axis a = 0.5 mpc, eccentricity e = 0.5, 0.9 respectively, and a period of approximately 15y, (Merrit et al., 2009). Using these values, we obtain numerical results that we tabulate in Table 1 below:

Table1 Secular time rate of change of the semimajor axis due to gravitational radiation when the secondary orbits around a super massive black hole.

Mass of the secondary m [kg]	Eccentricity e	Time rate of change of semimajor axis $(da/dt)_{e=e_1}$ [m/a]	Time rate of change of semimajor axis $(da/dt)_{e=e_2}$ [m/a]
3.3022×10^{23}	0.5 and 0.9	-9.800×10^{-7}	-0.00411
5.9742×10^{24}	0.5 and 0.9	-1.751×10^{-5}	-0.07345
1.8988×10^{27}	0.5 and 0.9	-0.00563	-23.6384
3.3022×10^{23}	0.999		-3.792×10^{7}
5.9742×10^{24}	0.999		-6.775×10^{8}
1.8988×10^{27}	0.999		-2.180×10^{11}

Table2 Maximum long frequency time rate of change of the semimajor axis due to gravitational radiation when the secondary orbits around a super massive black hole

Mass of the secondary m [kg]	Eccentricitye	Max semimajor axis effect [m]	Max semimajor axis effect [m]
3.3022×10^{23}	0.5 and 0.9	-3.807×10^{-13}	-1.658×10^{-8}
5.9742×10^{24}	0.5 and 0.9	-6.802×10^{-12}	-2.963×10^{-7}
1.8988×10^{27}	0.5 and 0.9	-2.189×10^{-9}	-9.534×10^{-5}
3.3022×10^{23}	0.999		-215.719
5.9742×10^{24}	0.999		-3854.22
1.8988×10^{27}	0.999		-1240×10^{6}

Next, we continue with the low frequency effect on the semimajor axis. Using Eq. (20) we obtain the maximum possible effect for Mercury to be to be -4.498×10^{-20} m. This effect is extremely small to be measured by today's Earth based technology. In the case of the SBH we tabulate the results in Table 2 below. Finally, for the numerical calculation of the high frequency effects of Eq (21) on the orbital time rate of change of the semimajor axis we choose to calculate only the maximum effect because Eq (22) contain many sine waves of various frequencies. We achieve this achieved by setting all trigonometric terms equal to unity, implying that all constituent waves are in phase. Substituting the numerical values given above in Eq. (23) we obtain the maximum (in phase) effect of the semimajor axis to be -8.740×10^{-20}. m. This effect is again extremely small to be measured by today's Earth based technology In the case of the SBH we tabulate the results in Table 3 below.

Table 3 Maximum high frequency time rate of change of the semimajor axis due to gravitational radiation when the secondary orbits around a super massive black hole

Mass of the secondary m [kg]	Eccentricities of the secondary e	Max change of semimajor axis [m]	Max change of semimajor axis [m]
3.3022×10^{23}	0.5 and 0.9	-3.993×10^{-13}	-8.989×10^{-10}
5.9742×10^{24}	0.5 and 0.9	-7.135×10^{-12}	-1.606×10^{-8}
1.8988×10^{27}	0.5 and 0.9	-2.296×10^{-9}	-5.168×10^{-6}

The higher the mass and the eccentricity of the primary the faster the semimajor axis will decay, and the faster it will crush into the SBH. A high eccentric orbit will bring the periastron closer to the SBH. For a small, the effect will have a greater effect at the vicinity of the periastron. In particular the enhancement factors $f(e)(1-e^2)^{-7/2}$ and $f(e)\ (1-e^2)^{-5}$are very sensitive to the changes of eccentricity. For example from the first one we obtain that f(0.5) = 4.88, f(0.6) = 10…. f(0.9) = 1243, f(0.99) = 3.9×10^6, f(0.999) = 1.237×10^{10}, and from the second one f(0.5) = 7.520, f(0.6) = 19.97…. f(0.9) = 15010, f(0.99) = 1.394×10^9, f(0.999) = 1.384×10^{14} respectively. Therefore, the highly eccentric orbits will suffer a great semimajor axis loss due also to the enhancement factor. In the case where e = 0.999, the secular effects reduce the orbit drastically in relatively small time, as given in Table (4).

In particular, the mass of Jupiter body will completes approximately 1.2 revolutions around the SBH before it crushes on to it.

Table 4 Orbital decaying times at different eccentricities

Mass of the secondary m [kg]	Eccentricities of the secondary e	Decaying time (secular effect) [y]	Max decaying time (low frequency effect) [y]	Max decaying time (high frequency effect) [y]
3.3022×10^{23}	0.5	1.872×10^{18}	1.872×10^{18}	2.883×10^{18}
5.9742×10^{24}	0.5	1.048×10^{17}	1.048×10^{17}	1.614×10^{17}
1.8988×10^{27}	0.5	3.256×10^{14}	3.256×10^{14}	5.014×10^{14}
3.3022×10^{23}	0.9	9.382×10^{14}	9.382×10^{14}	1.133×10^{16}
5.9742×10^{24}	0.9	5.251×10^{13}	5.251×10^{13}	6.340×10^{14}
1.8988×10^{27}	0.9	1.631×10^{11}	1.631×10^{11}	1.970×10^{12}
3.3022×10^{23}	0.999	101714	101714	1.138×10^{9}
5.9742×10^{24}	0.999	5692.89	5692.89	6.369×10^{7}
1.8988×10^{27}	0.999	17.6891	17.6891	197918

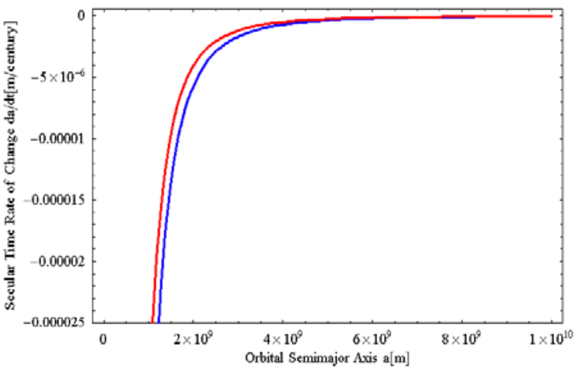

Fig 1 Secular semimajor axis time rate of Mercury caused by emission of gravitational radiation. Red line indicates a circular orbit and blue line an elliptical one.

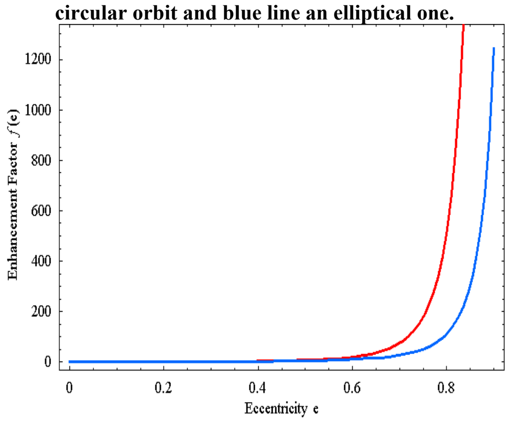

Fig 2 Enhancement factor f(e) versus eccentricity e. The red graph corresponds to secular effects and blue to high frequency effects,

From Fig. 2 above we observe that the enhancement functions for secular and high frequency effects are identical for values of eccentricities of approximately e = 0.4. For values higher than 0.4 the secular enhancement function drastically increases at around e = 0.65 where the corresponding high frequency one, increases at approximately e = 0.75. Therefore, secular effect are enhanced at lower eccentricity values comparing to the high frequency ones that are enhanced at slightly higher eccentricities, with an eccentricity difference of about 0.1.

CONCLUSIONS

We use the equation that gives the time rate of the orbital semimajor axis due to the emission of gravitational waves, in order to calculate secular, low, and high frequency effects of the semimajor axis decay. In addition, we used Kaula's approach to transform the inverse of the semimajor axis, in spherical harmonics, in order to consider the different cases mention in the above paragraph. We fist estimated the secular, low and high frequency effects on Mercury using Mercury and the Sun, and we found that the effects are very small to be measured with today's Earth based technology. The secular, low, and high frequency effects, on the time rate of change of the semimajor axis for three planetary bodies of different mass and high eccentricities orbiting a super massive black hole (SBH), were also calculated. From all effects, the secular time rate of change of the semimajor axis of the secondary of higher mass and eccentricity is the most significant for the orbital evolution of such a system. Even though all these effects are huge by Earth orbiting satellite standards, there may be small if such a system containing the SBH is observed from the Earth. In order to observe these effects we will probably have to resort in future space-orbiting technologies. Future technologies of this kind might prove suitable for detecting gravitational radiation of effects.

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